Numerical Process to Solve Two -Point Boundary Value Problems

G. Gangadhar[1], Dr.K. Sharath Babu[2] & Dr.V.Srinivas Kumar[3]

[1]*Research Scholar, JNTUH, Hyderabad, Telangana, India.*

[2]*Department of Science & Humanities,*

Matrusri Engineering College, Hydearbad, Telangana, India.

[3]*Department of Mathematics , JNTUH, Hyderabad, Telangana, India.*

[*]*Corresponding Author E-Mail sharathsiddipet@gmail.com*

Abstract: *Two point boundary value problems will occur in many fields of science and Engineering fields. With the advent of modernized technology like parallel computing, Algorithm analysis, development and its associate coding, many numerically approximate results one can generate. Similarly we made an attempt to solve a differential equation with numerical integration method approach. In the process Thomas algorithm is a powerful aid to get the problem simplification is nothing but back substitution of the entry values. Selected singularly perturbed differential equation numerically computed and compared its reliability with available closed form solutions and found this method is a reasonably good approximation method.*

Keywords: **Two point boundary value problem, Perturbation parameter, approximation method, Thomas algorithm.**

Introduction :

Two point boundary value problems can occur very frequently in various fields of Science and Engineering. Since closed form solutions for most of these problems are not available so one has to find to get the solutions of such problems in approximation methodology. The availability of high speed digital computations has made it possible to consider and solve such a work, when the chosen approximation method involves computation. The most frequently adoptable approximate methods for solving such problems are Finite difference Methods, numerical quadrature method and Finite volume Methods etc.

In this present research problem a two point boundary value problem with derivative boundary conditions are considered. Taylor's series approximation is used to reduce a second order differential equation into a first order differential equation subsequently the first order differential equation transformed into a finite difference equation. Numerical integration method and subsequently Thomas algorithm employed to get the approximate solution in the defined region at each mesh point for the various perturbation parameter values. It is observed that numerically computed solutions are in good agreement on par with the available closed form solutions.

For the sake of convenience we call our method the 'Numerical Integration Method'. To set the stage for the numerical integration method, we consider the following Governing linear Convection-diffusion two-point boundary value problem:

$$\varepsilon y''(x) + a(x) y'(x) + b(x)y(x) = g(x);\ 0 \le x \le 1 \qquad (1)$$

With y (0) =0 y $'(1) = \beta_1$, y(1) $=\beta$ (Derivative boundary conditions) (2)

β_1 value is only useful to fix the deviation parameter value approximately.

Where ε is a small positive parameter called diffusion parameter which lies in the interval $0<\varepsilon << 1$ i.e very close to zero, α and β are given constants; a(x) , b(x) and g(x) considered to be sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $a(x) \ge M > 0$ throughout the interval [0,1], where M is some positive constant. This assumption purely implies that the boundary layer will be in the neighborhood of x=0.

Let δ be a small positive deviating argument ($0< \delta \le 1$). By applying Taylor series expansions in the neighborhood of the point x, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \qquad (3)$$

and consequently, Eqn. (1) is replaced by the following first-order differential equation with a small deviating argument.

$$\frac{\delta^2}{2} y''(x) = y(x-\delta) - y(x) + \delta y'(x) \Rightarrow y''(x) = \frac{2}{\delta^2}[y(x-\delta) - y(x) + \delta y'(x)] \text{ So that}$$

$$(1) \Rightarrow \frac{2\varepsilon}{\delta^2}[y(x-\delta) - y(x) + \delta y'(x)] + a(x) y'(x) + b(x) y(x) = g(x),\ 0 \le x \le 1$$

$$\Rightarrow 2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\delta y'(x) + a(x)y'(x)\delta^2 + b(x)y(x)\delta^2 = \delta^2 g(x)$$

$$\Rightarrow [2\varepsilon\delta + a(x)\delta^2](y'(x)) + [b(x)\delta^2 - 2\varepsilon] y(x) = \delta^2 g(x) - 2\varepsilon y(x-\delta)$$

$$\Rightarrow y'(x) = \frac{\delta^2 g(x) - 2\varepsilon y(x-\delta)}{2\varepsilon\delta + a(x)\delta^2} y(x-\delta) + \frac{(2\varepsilon - b(x)\delta^2)}{2\varepsilon\delta + a(x)\delta^2} y(x)$$

$$\Rightarrow y'(x) = \frac{-2\varepsilon}{2\varepsilon\delta + a(x)\delta^2} y(x-\delta) + \frac{2\varepsilon - b(x)\delta^2}{2\varepsilon\delta + a(x)\delta^2} y(x) + \frac{\delta^2 g(x)}{2\varepsilon\delta + a(x)\delta^2} \qquad (4)$$

(4) can be re-written as

$$y'(x) = p(x) y(x-\delta) + q(x) y(x) + r(x) \text{ for } \delta \le x \le 1 \qquad (5)$$

Where $p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + a(x)\delta^2}$ (6)

$$q(x) = \frac{2\varepsilon - \delta^2\, b(x)}{2\,\varepsilon\delta + \delta^2\, a(x)} \quad (7)$$

$$r(x) = \frac{\delta^2\, g(x)}{2\,\varepsilon\delta + \delta^2\, a(x)} \quad (8)$$

We now divide the interval [0,1] into N equal parts with mesh size h, i.e., h=1/N and $x_i = ih$ for i= 1,2,3…….N. Integrating equation (5) in $[x_{i-1}, x_{i+1}]$ we get

$$y(x_{i+1}) - y(x_{i-1}) = \int_{x_{i-1}}^{x_{i+1}} [p(x)\, y(x-\delta) + q(x)y(x) + r(x)]\, dx \quad i = 1,2,3 \quad (9)$$

By implementing the Newton-Cotes approximate formula with n=2 i.e. by the virtue of Simpson's one-third rule , dividing the sub-intervals which are multiples of two then

$$y(x_{i+1}) - y(x_{i-1}) = \frac{h}{3}[p(x_{i+1})y(x_{i+1} - \delta) + 4\, p(x_i)\, y(x_i - \delta) + p(x_{i-1} - \delta)$$
$$+ (p_{i+1} + p_{i-1})\left[y(x_{i+1} - \delta) + y\left(x_{i-1} - \delta\right)\right] + q(x_{i+1})\, y(x_{i+1}) + q(x_{i-1})\, y(x_{i-1}) +$$
$$q(x_{i+1})y(x_{i+1}) + 4q(x_i)y(x_i) + q(x_{i-1})y(x_{i-1}) + r(x_{i+1}) + 4r(x_i) + r(x_{i-1}) +$$
$$r(x_{i+1}) + r(x_{i-1})\,] \quad (10)$$

Again by virtue of Taylor's series expansion

we have $y(x-\delta) \cong y(x) - \delta\, y'(x)$ Approximating $y'(x)$ by linear interpolation method

$$y(x_i - \delta) \cong y(x_i) - \frac{\delta[y(x_{i+1}) - y(x_{i-1})]}{2h}$$

$$= y(x_i) + \frac{\delta}{2h} y(x_{i-1}) - \frac{\delta}{2h} y(x_{i+1}) \quad (11)$$

Similarly $y(x_{i-1} - \delta) \cong (1 + \frac{\delta}{h})\, y(x_{i-1}) - \frac{\delta}{h} y(x_i)$ (12)

$$y(x_{i+1} - \delta) \approx (1 - \frac{\delta}{h})\, y(x_{i+1}) + \frac{\delta}{h} y(x_i) \quad (13)$$

Making use of (11),(12) and (13) in (10), it can be obtained as

$$A_i\, y_{i-1} + B_i\, y_i + C_i\, y_{i+1} = D_i \quad (14)$$

Where $A_i = -1 - \frac{2p_i\delta}{3} - \frac{h}{3} p_{i-1}(1 + \frac{\delta}{2h}) - \frac{h}{3}(p_{i+1} + p_{i-1})(1 + \frac{\delta}{h}) - \frac{2h}{3} q_{i-1}$ (15)

$$B_i = \frac{\delta\, p_{i-1}}{3} - \frac{\delta}{3} p_{i+1} - \frac{4hp_i}{3} - \frac{4hq_i}{3} \quad (16)$$

$$C_i = 1 - \frac{h}{3} p_{i+1}(1 - \frac{\delta}{h}) + \frac{2\, p_i \delta}{3} - \frac{h}{3}(p_{i+1} + p_{i-1})(1 - \frac{\delta}{h}) - \frac{2\, h}{3} q_{i+1} \quad (17)$$

$$D_i = \frac{2h}{3}[r_{i+1} + 2r_i + r_{i-1}] \quad (18)$$

Here $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Equation (14) gives a system of (n-1) equations with (n+1) unknown's y_0 to y_N. The two given boundary conditions (2) together with these (n-1) equations are in this case sufficient to solve for the unknowns y_0 to y_n. The solution of the Tri-diagonal system (14) can be obtained by using the efficient algorithm due to Thomas Algorithm. In this algorithm we set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \tag{19}$$

Where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined from (20)

we ha $y_{i-1} = W_{i-1} y_i + T_{i-1}$ (20)

Substituting (20) in (14) we have

$$y_i = \frac{C_i}{B_i - A_i W_{i-1}} y_{i+1} + \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \tag{21}$$

By comparing similar equations

$$W_i = \frac{C_i}{B_i - A_i W_{i-1}} \tag{22}$$

$$T_i = \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \tag{23}$$

To solve these recurrence relations for i=1,2, 3,…….N-1; we need to know the initial conditions for W_0 and T_0. This can be done by considering (2) function value can be calculated with the help of interpolation.

$$y_0 = \alpha = W_0 y_1 + T_0 \tag{24}$$

If we choose $W_0 = 0$, then $T_0 = \alpha$. With these initial values , we compute sequentially W_i and T_i for i=1,2,3,….N-1;from (23) and (24) in the forward process and then obtain y_i in the backward process from (19) using the equation (2).

Repeat the numerical scheme for different choices of δ (deviating argument) satisfying the conditions $(0 < \delta \le \beta_1)$, until the solution profiles do not differ significantly from iteration to iteration. So that initially we are selected small value of deviating argument For computational point of view and convergent solution.

We use an absolute error criterion, $|y(x)^{m+1} - y(x)^m| \le \rho, 0 \le x \le 1$ (25)

Where $y(x)^m$ the solution for the m^{th} iterate of δ, and ρ is the prescribed tolerance bound considered.

COMPUTED PROBLEMS

For testing the applicability of the numerical integration method; we have applied it to linear singular perturbation problems with left-end boundary layer. These examples have been chosen. Reason behind this is they have been widely discussed in the literature and approximate solutions are available for comparison.

Example 1:

Consider the following homogeneous singularly perturbed problem from Kevorkian and Cole [36], p.33, Eq. (2.3.26) and (2.3.27) with α =0:

$\varepsilon\, y''(x) + y'(x) = 0$, $0 \le x \le 1$ with y(0)=0 and y(1)=1 . The exact solution is given by analytical method $y(x) = \frac{(1-e^{(-x/\varepsilon)})}{(1-e^{(-1/\varepsilon)})}$

The computational results are presented in Table 1(a) and 1(b) for $\varepsilon = 10^{-3}, 10^{-4}$ respectively with the known β_1 value which selects δ approximately which is very close to zero. As δ is small so that we can observe more accuracy.

Computational results for Example 1.

(a) **$\varepsilon=10^{-3}$, h=0.01.**

Table 1(a)

x	y(x)			Exact solution
	δ=0.008	δ=0.009	δ=0.01	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9876486	0.9899944	0.9917358	1.0000000
0.04	0.9998419	0.9998944	0.9999319	1.0000000
0.06	0.9999925	0.9999934	0.9999995	1.0000000
0.08	0.9999945	0.9999945	1.0000000	1.0000000
0.10	0.9999946	0.9999948	1.0000000	1.0000000
0.20	0.9999954	0.9999952	1.0000000	1.0000000
0.40	0.9999964	0.9999964	1.0000000	1.0000000
0.60	0.9999976	0.9999976	1.0000000	1.0000000
0.80	0.9999988	0.9999988	1.0000000	1.0000000
1.00	1.00000000	1.00000000	1.0000000	1.0000000

(b)ε= 10^{-4} and h= 0.01

Table.1 (b)

x	y(x)			Exact solution
	δ=0.007	δ=0.008	δ-0.009	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9998016	0.9998477	0.9998792	1.0000000
0.04	0.9999999	1.0000000	1.0000000	1.0000000
0.06	1.0000000	1.0000000	1.0000000	1.0000000
0.08	1.0000000	1.0000000	1.0000000	1.0000000
0.10	1.0000000	1.0000000	1.0000000	1.0000000
0.20	1.0000000	1.0000000	1.0000000	1.0000000
0.40	1.0000000	1.0000000	1.0000000	1.0000000
0.60	1.0000000	1.0000000	1.0000000	1.0000000
0.80	1.0000000	1.0000000	1.0000000	1.0000000
1.00	1.00000000	1.00000000	1.0000000	1.0000000

Example 2 :

Consider the following homogeneous Singular perturbation problem from Bender and Orsag [10] ,p.480. Problem 9.17 with α =0:

$$\varepsilon y''(x) + y'(x) - y(x) = 0 \quad 0 \le x \le 1 \text{ with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by $y(x) = \dfrac{(e^{m_2} - 1)\, e^{m_1 x} + (1 - e^{m_1})\, e^{m_2 x}}{(e^{m_2} - e^{m_1})}$

Where $m_1 = \dfrac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$; $m_2 = \dfrac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$

Computational results for Example 2 are furnished in table **2(a)** and **2(b)**.

Computational results for Example.2

(a) $\varepsilon=10^{-3}$, h=0.01. **Table. 2(a)**

x	y(x)			Exact solution
	δ=0.008	δ=0.009	δ=0.01	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3834784	0.3819605	0.3808348	0.3756784
0.04	0.3834410	0.3833556	0.3832939	0.3832599
0.06	0.3910826	0.3910290	0.3909866	0.3909945
0.08	0.3989720	0.3989188	0.3988770	0.3988851
0.10	0.4070216	0.4069688	0.4069269	0.4069350
0.20	0.4497731	0.4497210	0.4496799	0.4496879
0.40	0.5492185	0.5491707	0.5491330	0.5491404
0.60	0.6706514	0.6706123	0.6705816	0.6705877
0.80	0.8189330	0.8189092	0.8188905	0.8188942
1.00	1.0000000	1.0000000	1.0000000	1.0000000

(b) $\varepsilon= 10^{-4}$ and h= 0.01:

Table.2 (b)

x	y(x)			Exact solution
	δ=0.007	δ=0.008	δ=0.009	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3754246	0.3754509	0.3754841	0.3753479
0.04	0.3829308	0.3829373	0.3829417	0.3829296
0.06	0.3906657	0.3906722	0.3906766	0.3906645
0.08	0.3985569	0.3985633	0.3985675	0.3985557
0.10	0.4066074	0.4066138	0.4066185	0.4066062
0.20	0.4493662	0.4493724	0.4493767	0.4493649
0.40	0.5488456	0.5488514	0.5488553	0.5488445
0.60	0.6703477	0.6703524	0.6703555	0.6703469
0.80	0.8187476	0.8187505	0.8187521	0.8187471
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Example 3

Consider the following non-homogeneous Singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad 0 \le x \le 1 \text{ with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by $y(x) = x(x + 1\text{-}2\varepsilon) + (2\varepsilon\text{-}1)\dfrac{(1\text{-}\exp(\text{-}x/\varepsilon))}{(1\text{-}\exp(\text{-}1/\varepsilon))}$

The computational results are presented in Table 3(a) and 3(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

(a) $\varepsilon= 10^{-4}$ and h=0.01 Table 3(a)

x	y(x)			Exact solution
	δ=0.009	δ=0.008	δ=0.007	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.9648339	-0.9674433	-0.9693918	-0.9776401
0.04	-0.9558469	-0.9561648	-0.9564114	-0.9564800
0.06	-0.9340471	-0.9343091	-0.9345188	-0.9345200
0.08	-0.9112990	-0.9115545	-0.9117596	-0.9117600
0.10	-0.8877492	-0.8879992	-0.8881995	-0.8882000
0.20	-0.7579996	-0.7582219	-0.7583995	-0.7584000
0.40	-0.4385004	-0.4386670	-0.4387995	-0.4388000
0.60	-0.0390007	-0.0391119	-0.0391996	-0.0391999
0.80	0.4404994	0.4404438	0.4404002	0.4404000
1.00	1.0000000	1.00000000	1.00000000	1.00000000

(b) ε=0.001, h=0.01 Table 3(b)

X	y(x)			Exact solution
	δ=0.008	δ=0.009	δ=0.01	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.9794212	-0.9792020	-0.9792610	-0.9794040
0.04	-0.6581250	-0.9581596	-0.9581869	-0.9582080
0.06	-0.9361311	-0.9361844	-0.9361909	-0.9362120
0.08	-0.9133368	-0.9133694	-0.9133958	-0.9134160
0.10	-0.8897421	-0.8897744	-0.8897998	-0.8898200
0.20	-0.7597710	-0.7597994	-0.7598217	-0.7598400
0.40	-0.4398281	-0.4398495	-0.4398661	-0.4398800
0.60	-0.0398852	-0.0398996	-0.0399109	-0.0399199
0.80	0.4400563	0.4400498	0.4400447	0.4400400
1.00	1.00000000	1.00000000	1.00000000	1.00000000

Discussion &Conclusion:

In contrast with the direct methods, numerical methods are not expected to terminate in a finite number of steps, starting from an initial approximate solution if available. An approximate solution is as near as to the exact solution. But there is a possibility of an error. Our main aim in the numerical analysis is to reduce the possibility of error in all the aspects so that there is a good equilibrium with the numerically computed solution and closed form solution.

Numerical Integration method implemented for a two point boundary value problem. After deriving the calculations we are observed that numerically computed values are shown to be good approximation to the selected problems and also the values which are compared with the closed form solutions available in the literature. It is observed that the numerically obtained results are in good agreement with the analytical solutions with reasonable accuracy.

REFERENCES:

[1]. Carl M. Bender ,Steven A. orszag Advanced Mathematical methods for Scientists and Engineers, Asymptotic methods and perturbation theory, Springer.

[2]. Elementary Differential Equations and Boundary Value Problemsby William E. Boycem Wiley International edition.

[3]. Introduction to singular Perturbation problems by Robert E.O"s Malley,Jr, Academic press.

[4]. L.E. El'sgol'ts, S.B.Norkin, Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press, New York, (1973).

[5]. Keller, Herbert B, Numerical methods for two point boundary value problemsDover publications.

[6]. K.Eriksson, D. Estep, P.Hansbo and C. Johnson (1996) , Computational DifferentialEquations, Cambridge University Press, Cambridge.

[7]. M. Stynes and L. Tobiska (1998),A finite difference analysis of a streamline diffusion method on a Shishkin meshes", Numer. Algorithms 18, 337360.

[8]. R.E.O'Malley, Singular Perturbation Methods for Ordinary Differential Equations, Springer, New York, (1991).